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Discrete-Time Output Trajectory Tracking by Recurrent High-Order Neural Network Control

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Abstract— **This paper presents the design of an adaptive controller based on the block control technique, and a new neural observer for a class of MIMO discretr-time nonlinear systems. The observer is based on a recurrent high-order neural network (RHONN), which estimates the state vectors of the unknown plant dynamics. The learning algorithm for the RHONN is based on an Extended Kalman Filter (EKF). This paper also includes the respective stability analysis, from the Lyapunov approach, for the neural observer trained with the EKF and the block controller. Simulation results are included to illustrate the applicability of the proposed scheme.**

Keywords— **Sliding mode, Recurrent high-order neural network, Extended Kalman filter, Nonlinear observer, Discrete-time block control.**

I. Introduction

Nonlinear trajectory tracking is an important research subject ([1], [8], [9], [10], and some references cited therein; mostly for continuous-time systems). In the recent literature on adaptive and robust controls, numerous approaches have been proposed for nonlinear control system trajectory tracking, among which the block control strategy provides a well-suited design methodology [5]. In most nonlinear control designs, it is usually assumed that all the system states are measurable. In practice, however, only parts of these states are measured directly. For this reason, nonlinear state estimation remains an important topic for study in the nonlinear systems theory [11]. Numerous approaches have been proposed for the design of nonlinear observers, yielding many interesting results in different directions ([7], [11] and references therein). Most of these approaches in developing nonlinear observers require a nominal mathematical model of the plant dynamics, if not fully then at least partially [11]. Recurrent neural-network observers have also been proposed, and they do not require a precise plant model. This technique is therefore attractive and actually has been successfully applied to state estimation [11], [13]. These works were developed mostly for continuous-time systems. Nonlinear discrete-time neural observers, on the other hand, have been seldom discussed [13].

The best-known training approach for recurrent neural networks (RNN) is the back propagation through time learning [16]. However, it is merely a first-order gradient descent method and hence its learning speed is very slow [16]. Recently, some Extended Kalman Filter (EKF) based algorithms have been introduced to the training of neural networks [16]. With an EKF-based algorithm, the learning convergence can be improved [16]. Over the past decade, the EKF-based training of neural networks, both feedforward and recurrent ones, has proven to be reliable and practical for many applications [16].

In this paper, we propose a scheme for trajectory tracking based on the block control technique [5], using a new neural observer for a class of MIMO discrete-time nonlinear systems. This observer is based on a discrete-time recurrent high-order neural network (RHONN), which estimates the state vectors of the unknown plant dynamics. The learning algorithm for the RHONN is based on an EKF. Once the neural network structure is determined, the block control technique is used to develop the corresponding trajectory tracking controller. This paper also includes the respective stability analysis, from the Lyapunov approach, for the neural observer trained with the EKF and the block controller. Finally, the applicability of the proposed design is illustrated by a typical example: output chaos synchronization of discrete-time systems.

II. Mathematical Preliminaries

Let k denote the sampling step, $k \in 0 \cup \mathbb{Z}^{+}$, $|\cdot|$ be the absolute value, and $\|\cdot\|$ be the Euclidian norm for vectors and an adequate norm for matrices.

Following [6], consider an MIMO nonlinear system,

$$x(k+1) = F(x(k), u(k)) \tag{1}$$

where $x \in \Re^{n}$, $u \in \Re^{m}$, and $F \in \Re^{n} \times \Re^{m} \to \Re^{n}$ is a nonlinear map.

In system (1), after selecting the input u as a feedback function of the state:

$$u(k) = h(x(k)) \tag{2}$$

one can obtain

$$x(k+1) = F(x(k), h(x(k))) \tag{3}$$

which yields an unforced system

$$x\left(k+1\right)=\widetilde{F}\left(x\left(k\right)\right) \tag{4}$$

Now, consider the following discrete-time RHONN:

$$\widehat{x}_i(k+1)=w_i^\top z_i(\widehat{x}(k),u(k)),\quad i=1,\cdots,n \tag{5}$$

where $\widehat{x}_i$ $(i=1,2,\cdots,n)$ is the state of the ith neuron, L_i is the respective number of higher-order connections, $\{I_1,I_2,\cdots,I_{L_i}\}$ is a collection of non-ordered subsets of $\{1,2,\cdots,n\}$, n is the state dimension, w_i $(i=1,2,\cdots,n)$ is the respective on-line adapted weight vector, and $z_i(\widehat{x}(k),u(k))$ is given by

$$z_i(x(k),u(k))=\begin{bmatrix} z_{i_1}\\ \vdots \\ z_{i_{L_i}}\end{bmatrix}=\begin{bmatrix}\Pi_{j\in I_1}y_{i_j}^{di_j(1)}\\ \vdots \\ \Pi_{j\in I_{L_i}}y_{i_j}^{di_j(L_i)}\end{bmatrix} \tag{6}$$

with $d_{j_i}(k)$ being a nonnegative integers, and

$$y_i=\begin{bmatrix} y_{i_1}\\ \vdots \\ y_{i_1}\\ y_{i_{n+1}}\\ \vdots \\ y_{i_{n+m}}\end{bmatrix}=\begin{bmatrix} S(\widehat{x}_1)\\ \vdots \\ S(\widehat{x}_n)\\ S\left(u_1\right)\\ \vdots \\ S\left(u_m\right)\end{bmatrix} \tag{7}$$

in which $u=[u_1,u_2,\ldots,u_m]^\top$ is the input vector to the neural network, and $S(\cdot)$ is defined by

$$S(x)=\frac{1}{1+\exp(-b\widehat{x})} \tag{8}$$

where $b>0$ is a constant.

Consider the problem of approximating the general discrete-time nonlinear system (1), which is supposed to be observable, by the following discrete-time RHONN parallel representation [14]:

$$x_i\left(k+1\right)=w_i^{*\top}z_i\left(x(k),u(k)\right)+\epsilon_{z_i} \tag{9}$$

where x_i is the ith plant state, ϵ_{z_i} is a bounded approximation error, which can be reduced by increasing the number of the adjustable weights [14].

Assume that there exists ideal weights vector w_i^* such that $\|\epsilon_{z_i}\|$ can be minimized on a compact set $\Omega_{z_i}\subset\Re^{L_i}$. The ideal weight vector w_i^* is an artificial quantity required for analysis [14]. In general, it is assumed that this vector exists and is an unknown constant. Define its estimate as w_i and the estimation error as

$$\widetilde{w}_i\left(k\right)=w_i\left(k\right)-w_i^* \tag{10}$$

The estimate w_i is used for stability analysis, which will be discussed later. Since w_i^* is constant, one has

$$\widetilde{w}_i\left(k+1\right)-\widetilde{w}_i\left(k\right)=w_i\left(k+1\right)-w_i\left(k\right)$$

III. The EKF training algorithm

The Kalman filter (KF) estimates the state of a linear system with additive state and output white noises [4], [7], [15]. For KF-based neural network training, the network weights become the states to be estimated. In this case, the error between the neural network output and the measured plant output can be considered as additive white noise. Since the neural network mapping is nonlinear, an EKF is applied (see [12] and references therein). The training goal is to find the optimal weight values that minimize the prediction errors.

In this paper, we use a decoupled EKF-based training algorithm described by

$$\begin{aligned} w_i\left(k+1\right) &= w_i\left(k\right)+\eta_i K_i\left(k\right)e\left(k\right),\quad i=1,\cdots,n\\ K_i\left(k\right) &= P_i\left(k\right)H_i\left(k\right)M_i\left(k\right) \\ P_i\left(k+1\right) &= P_i\left(k\right)-K_i\left(k\right)H_i^\top\left(k\right)P_i\left(k\right)+Q_i\left(k\right)\end{aligned} \tag{11}$$

with

$$\begin{aligned} M_i\left(k\right) &= \left[R_i\left(k\right)+H_i^\top\left(k\right)P_i\left(k\right)H_i\left(k\right)\right]^{-1} && (12)\\ e\left(k\right) &= y\left(k\right)-\widehat{y}\left(k\right) && (13)\end{aligned}$$

where $e\left(k\right)\in\Re^p$ is the observation error and $P_i\left(k\right)\in\Re^{L_i\times L_i}$ is the prediction error covariance matrix at step k, $w_i\in\Re^{L_i}$ is the weight (state) vector, L_i is the respective number of neural network weights, $y\in\Re^p$ is the plant output, $\widehat{y}\in\Re^p$ is the observer output, n is the number of states, $K_i\in\Re^{L_i\times p}$ is the Kalman gain matrix, $Q_i\in\Re^{L_i\times L_i}$ is the NN weight estimation noise covariance matrix, $R_i\in\Re^{p\times p}$ is the error noise covariance, and $H_i\in\Re^{L_i\times p}$ is a matrix, in which each entry (H_{ij}) is the derivative of the neural output with respect to one neural network weight, (w_{ij}), given as follows:

$$H_{ij}\left(k\right)=\left[\frac{\partial\widehat{y}\left(k\right)}{\partial w_{i_j}\left(k\right)}\right]^\top_{w_i(k)=w_i(k+1)} \tag{14}$$

where $j=1,...,L_i$ and $i=1,...,n$. Usually, $P_i$and Q_i are initialized as diagonal matrices, with entries $P_i\left(0\right)$ and $Q_i\left(0\right)$, respectively. It is important to remark that $H_i\left(k\right)$, $K_i\left(k\right)$ and $P_i\left(k\right)$ for the EKF to be bounded (for a detailed explanation, see [15]).

IV. Neural Observer Design

In this section, we consider the estimation of the states of an observable discrete-time nonlinear system given by

$$\begin{aligned} x\left(k+1\right) &= F\left(x\left(k\right),u\left(k\right)\right)+d\left(k\right)\\ y\left(k\right) &= Cx\left(k\right)\end{aligned} \tag{15}$$

where $x\in\Re^n$ is the state vector of the system, $u\left(k\right)\in\Re^m$ is the input vector, $y\left(k\right)\in\Re^p$ is the output vector, $C\in\Re^{p\times n}$ is a known output matrix, $d\left(k\right)\in\Re^n$ is a disturbance vector, and $F\left(\cdot\right)$ is a smooth vector field with entries $F_i(\cdot)$. Hence, (15) can be rewritten as:

$$\begin{aligned} x(k) &= \left[\begin{array}{ccccc} x_1(k) & \ldots & x_i(k) & \ldots & x_n(k) \end{array}\right]^\top \\ d(k) &= \left[\begin{array}{ccccc} d_1(k) & \ldots & d_i(k) & \ldots & d_n(k) \end{array}\right]^\top \\ x_i(k+1) &= F_i(x(k), u(k)) + d_i(k) \ , \ i = 1, \cdots, n \\ y(k) &= Cx(k) \end{aligned} \tag{16}$$

For system (16), a Luenberger neural observer is used, with the following structure:

$$\begin{aligned} \widehat{x}(k) &= \left[\begin{array}{ccccc} \widehat{x}_1(k) & \ldots & \widehat{x}_i(k) & \ldots & \widehat{x}_n(k) \end{array}\right]^\top \\ \widehat{x}_i(k+1) &= w_i^\top z_i(\widehat{x}(k), u(k)) + L_i e(k) \\ \widehat{y}(k) &= C\widehat{x}(k), \quad i = 1, \cdots, n \end{aligned} \tag{17}$$

with $L_i \in \Re^p$, w_i and z_i as in (5). The weight vectors are updated on-line with a decoupled EKF $(11)-(14)$, and the output error is defined by

$$e(k) = y(k) - \widehat{y}(k) \tag{18}$$

and the state estimation error is

$$\widetilde{x}(k) = x(k) - \widehat{x}(k) \tag{19}$$

Before proceeding to demonstrating the main result of this paper, we need to establish the following two lemmas.

Lemma 1. Error (18) can be reformulated as

$$\begin{aligned} e(k) &= \left[\begin{array}{ccccc} e_1(k) & \cdots & e_j(k) & \cdots & e_p(k) \end{array}\right] \\ e_j(k+1) &= e_j(k) + \Delta e_j(k), \quad j = 1, \cdots, p \end{aligned} \tag{20}$$

with $\Delta e_j(k) \leq -\gamma_{i,j} e_j(k)$, $\gamma_{i,j} = \min_k \left|\eta_i H_{i,j}^\top(k) K_{i,j}(k)\right|$, $e_j(k)$ is the jth element of $e(k)$, $H_{i,j}(k)$ is the jth column of $H_i(k)$, and $K_{i,j}(k)$ is the jth column of $K_i(k)$.

Proof: It follows from (18) that

$$\frac{\partial e_j(k)}{\partial w_i(k)} = -\frac{\partial \widehat{y}_j(k)}{\partial w_i(k)} \tag{21}$$

Approximate (21) by

$$\Delta e_j(k) = \left[\frac{\partial e_j(k)}{\partial w_i(k)}\right]^\top \Delta w_i(k) \tag{22}$$

Substituting (14), (18) and (21) into (22) yields

$$\Delta e_j(k) = -\eta_i H_{i,j}^\top(k) K_{i,j}(k) e_j(k) \tag{23}$$

Define

$$\gamma_{i,j} = \min_k \left|\eta_i H_{i,j}^\top(k) K_{i,j}(k)\right|$$

Then

$$\Delta e_j(k) \leq -\gamma_{i,j} e_j(k) \tag{24}$$

■

Comment 1. From $(11)-(12)$, it follows that $H_{i,j}^\top(k) K_{i,j}(k) \geq 0$, $\forall k \in 0 \cup \mathbb{Z}^+$.

Lemma 2. The weight estimation error (10) can be written as

$$\begin{aligned} \widetilde{w}_i^\top(k) &= \left[\epsilon_i^{'}(k) - L_i e(k) - C_i^+ e(k) - C_i^+ \Delta e(k)\right] \\ &\quad \times z_i^+(x(k), u(k)) \end{aligned}$$

with C_i^+ being the ith row of C^+ and $\epsilon_i^{'}(k) = \epsilon_i + d_i(k)$.

Proof: It follows from $(16)-(18)$ that

$$\begin{aligned} e(k) &= y(k) - \widehat{y}(k) = C(x(k) - \widehat{x}(k)) \qquad (25) \\ e(k+1) &= C\left(w^{*\top} z(x(k), u(k)) + \epsilon + d(k)\right) \\ &\quad - C\left(w_i^\top(k) z_i(\widehat{x}(k), u(k)) + Le(k)\right) \\ &= e(k) + \Delta e(k) \end{aligned}$$

Hence,

$$\begin{aligned} \widetilde{w}^\top(k) z(x(k), u(k)) &= \epsilon + d(k) - Le(k) \\ &\quad - C^+(e(k) + \Delta e(k)) \end{aligned}$$

and

$$\begin{aligned} \widetilde{w}^\top(k) &= \left[\epsilon + d(k) - Le(k) - C^+ e(k) - C^+ \Delta e(k)\right] \\ &\quad \times z^+(x(k), u(k)) \end{aligned}$$

So, for the ith element, we have

$$\begin{aligned} \widetilde{w}_i^\top(k) &= \left[\epsilon_i^{'}(k) - L_i e(k) - C_i^+ e(k) - C_i^+ \Delta e(k)\right] \\ &\quad \times z_i^+(x(k), u(k)) \end{aligned} \tag{26}$$

■

By summarizing (11)-(19), we obtain the first main result as follows.

Theorem 1: For system (16), the nonlinear observer (17) trained with the EKF-based algorithm (11), ensures that the output error (18) and the estimation error (19) are semi-globally uniformly ultimately bounded.

Proof: Consider the Lyapunov function candidate, for $e(k)$, $w(k)$, defined by

$$V_i(k) = e^\top(k) e(k) + \widetilde{w}_i^\top(k) \widetilde{w}_i(k) \tag{27}$$

whose first difference is:

$$\begin{aligned} \Delta V_i(k) &= V_i(k+1) - V_i(k) \\ &= e^\top(k+1) e(k+1) + \widetilde{w}_i^\top(k+1) \widetilde{w}_i(k+1) \\ &\quad - e^\top(k) e(k) - \widetilde{w}_i^\top(k) \widetilde{w}_i(k) \end{aligned} \tag{28}$$

From (10) and (11), we have

$$\widetilde{w}_i(k+1) = \widetilde{w}_i(k) + \eta_i K_i(k) e(k) \tag{29}$$

Define

$$\begin{aligned} &\left[\widetilde{w}_i(k) + \eta_i K_i(k) e(k)\right]^\top \\ &\times \left[\widetilde{w}_i(k) + \eta_i K_i(k) e(k)\right] \\ = \ &\widetilde{w}_i^\top(k) \widetilde{w}_i(k) + 2\eta_i \widetilde{w}_i^\top(k) K_i(k) e(k) \\ &+ (\eta_i K_i(k) e(k))^\top (\eta_i K_i(k) e(k)) \end{aligned} \tag{30}$$

Then, it follows from(18) that

$$\begin{aligned} e(k+1) &= e(k) + \Delta e(k) \\ e^\top(k+1)\,e(k+1) &= e^\top(k)\,e(k) + e^\top(k)\,\Delta e(k) \\ &+\Delta e^\top(k)\,e(k) + \Delta e^\top(k)\,\Delta e(k) \end{aligned} \quad (31)$$

Applying (30), (31) and Lemma 2 to (28), we have

$$\begin{aligned} \Delta V_i(k) &= e^\top(k)\,\Delta e(k) + \Delta e^\top(k)\,e(k) \\ &+\Delta e^\top(k)\,\Delta e(k) \\ &+2\eta_i\left[\epsilon_i - L_i e(k) - C_i^+ e(k) - C_i^+\Delta e(k)\right] \\ &\times z_i^+(k))\,K_i(k)\,e(k) \\ &+\left(\eta_i K_i(k)\,e(k)\right)^\top \eta_i K_i(k)\,e(k) \end{aligned} \quad (32)$$

From Lemma 1, substituting (24) into the above, with $\delta_i = \eta_i\gamma_i C_i^+ - L_i - C_i^+$ and $\gamma_i = \max\{\gamma_{i,j}\}$, Eq. (32) can be written as

$$\begin{aligned} \Delta V_i(k) &\le -2\gamma_i\left\|e(k)\right\|^2 + \gamma_i^2\left\|e(k)\right\|^2 \\ &+2\left|\eta_i\epsilon_i z_i^+(k))\,K_i(k)\right|\left\|e(k)\right\| \\ &+2\left|\eta_i\delta_i z_i^+(k))\,K_i(k)\right|\left\|e(k)\right\|^2 \\ &+\left\|\eta_i K_i(k)\right\|^2\left\|e(k)\right\|^2 \end{aligned}$$

So, there exits $\eta_i > 0$ and L_i such that

$$\Delta V_i(k) < 0, \quad \text{for} \quad \left\|e(k)\right\| > \kappa_i \quad (33)$$

with

$$\kappa_i = \frac{\left|2\eta_i\epsilon_i\left\|z_i^+ K\right\|\right|}{\left|2\gamma_i - \gamma_i^2 - 2\eta_i\left\|\delta_i z_i^+ K_i\right\| - \eta_i^2\left\|K_i\right\|^2\right|}$$

From (33), it follows the boundness of $V_i(k)$ for a $k \ge k_T$, which leads to the semi-global uniformly ultimately boundedness of $e(k)$ and $\widetilde{w}_i(k)$.

Considering (19) and (25), it is easy to see that the estimation error has an algebraic relation with $e(k)$, so if $e(k)$ is bounded then $\widetilde{x}(k)$ is bounded too:

$$\begin{aligned} \widetilde{x}(k) &= C^+ e(k) \\ \left\|\widetilde{x}(k)\right\| &\le \left\|C^+\right\|\left\|e(k)\right\| \end{aligned} \quad (34)$$

Comment 2. Inequality (34) is established for the norm of $\widetilde{x}(k)$, which implies that $\widetilde{x}_i(k)$, $i = 1, \cdots, n$, is bounded.

Using (26) and (33), it is easy to see that the weight estimation error (10) has an algebraic relation with $e(k)$ and $z_i^+(x(k), u(k))$, so if $e(k)$ is bounded, and given that $z_i^+(x(k), u(k))$ as defined in (6) is bounded, then $\widetilde{w}(k)$ is bounded too:

$$\begin{aligned} \left|\widetilde{w}_i^\top(k)\right| &\le \left|\epsilon_i'(k) - L_i e(k) - C_i^+ e(k) - C_i^+\Delta e(k)\right| \\ &\times\left\|z_i^+(x(k), u(k))\right\| \end{aligned}$$

■

V. Controller Design

Consider a special case of system (15):

$$\begin{aligned} x(k+1) &= f(x(k)) + B(x(k))\,u(k) + d(k) \\ y(k) &= Cx(k) \end{aligned} \quad (35)$$

where $x \in \Re^n$ is the state vector of the system, $u(k) \in \Re^m$ is the input vector, $y(k) \in \Re^p$ is the output vector, the vector $f(\cdot)$, the columns of $B(\cdot)$ and $d(\cdot)$ are smooth vector fields, and $d(\cdot)$ is a disturbance vector. By means of non-singular transformation, system (15) can be represented in the block controllable form as follows:

$$\begin{aligned} x_i(k+1) &= f_i(\overline{x}_i(k)) + B_i(\overline{x}_i(k))\,x_{i+1}(k) + d_i(k) \\ x_r(k+1) &= f_r(x(k)) + B_r(x(k))\,u(k) + d_r(k) \\ y(k) &= x_1(k),\ i = 1, \cdots, r-1 \end{aligned} \quad (36)$$

where $x(k) = \begin{bmatrix} x_1(k) & \dots & x_i(k) & \dots & x_r(k) \end{bmatrix}^\top$, $d(k) = \begin{bmatrix} d_1(k) & \dots & d_i(k) & \dots & d_r(k) \end{bmatrix}^\top$ and $\overline{x}_i(k) = \begin{bmatrix} x_1(k) & \dots & x_i(k) \end{bmatrix}^\top$, $i = 1, \cdots, r-1$, and the set of numbers $(n_1, \cdots, n_r)$, which define the structure of system (36), satisfy $n_1 \le n_2 \le \cdots \le n_r \le m$.

Define the following transformation:

$$\begin{aligned} z_1(k) &= x_1(k) - x_d(k) \\ z_2(k) &= x_2(k) - \left[B_1(x_1(k))\right]^{-1}(K_1 z_1(k)) \\ &+\left[B_1(x_1(k))\right]^{-1}(f_1(x_1(k)) + d_1(k)) \\ &\vdots \\ z_r(k) &= x_r(k) - x_r^d(k) \end{aligned} \quad (37)$$

with $y_d(k) = x_d(k)$ as the desired trajectory for tracking. Using (37), system (36) can be rewritten as

$$\begin{aligned} z_1(k+1) &= K_1 z_1(k) + B_1 z_2(k) \\ &\vdots \\ z_{r-1}(k+1) &= K_{r-1} z_{r-1}(k) + B_{r-1} z_r(k) \\ z_r(k+1) &= f_r(x(k)) + B_r(x(k))\,u(k) \\ &+d_r(k) - x_r^d(k) \end{aligned} \quad (38)$$

To design the control law, we use the sliding mode block control technique. The surface is derived from the block control procedure, and a natural selection for the sliding surface $S(k) = 0$ is $S(k) = z_r(k) = 0$. Thus, system (38) is represented as

$$\begin{aligned} z_1(k+1) &= K_1 z_1(k) + B_1 z_2(k) \\ &\vdots \\ z_{r-1}(k+1) &= K_{r-1} z_{r-1}(k) + B_{r-1} S(k) \\ S(k+1) &= f_r(x(k)) + B_r(x(k))\,u(k) \\ &+d_r(k) - x_r^d(k) \end{aligned} \quad (39)$$

Once the sliding surface is selected, the next step is to define $u(k)$, as

$$u(k)=\begin{cases} u_{eq}(k) & \text{for} \quad \|u_{eq}(k)\|\le u_0 \\ u_0\frac{u_{eq}(k)}{\|u_{eq}(k)\|} & \text{for} \quad \|u_{eq}(k)\|> u_0 \end{cases} \tag{40}$$

where the equivalent control is calculated from $S(k+1)=0$, as

$$\begin{aligned} u_{eq}(k) &= [B_r(x(k))]^{-1} \\ &\quad \times(-f_r(x(k))+x_{r,d}(k+1)-d_r(k)) \end{aligned}$$

To this end, we present an stability analysis to prove that the closed-loop system motion over the surface is stable, which is the second main result of this paper.

Theorem 2. The control law (40) ensures the sliding surface $S(k)=z_r(k)=0$ for system (36).

Proof: Write $S(k+1)$ as

$$\begin{aligned} S(k+1)-S(k) &= f_r\left(x^1(k)\right)+B_r u(k) \\ &\quad +x_r^d(k)-x_r(k) \end{aligned}$$

Note that when $\|u_{eq}(k)\|\le u_0$, the equivalent control is applied, yielding motion on the sliding manifold $S(k)=0$. In the case of $\|u_{eq}(k)\|>u_0$, the proposed control strategy is $u_0\frac{u_{eq}(k)}{\|u_{eq}(k)\|}$, and the closed-loop system is

$$\begin{aligned} S(k+1) &= \left(S(k)+f_3\left(x^1(k)\right)-x^{2d}(k)+x^2(k)\right) \\ &\quad \times\left(1-\frac{u_0}{\|u_{eq}(k)\|}\right) \end{aligned}$$

Along any solution of the system, the Lyapunov candidate function $V(k)=S^\top(k)S(k)$ gives

$$\begin{aligned} \Delta V(k) &= S^\top(k+1)S(k+1)-S^\top(k)S(k) \\ &= \left[(S(k)+f_s(k))\left(1-\frac{u_0}{\|u_{eq}(k)\|}\right)\right]^\top \\ &\quad \times(S(k)+f_s(k))\left(1-\frac{u_0}{\|u_{eq}(k)\|}\right) \\ &\quad -S^\top(k)S(k) \\ &\le -2\|S(k)\|\left(\frac{u_0}{\left\|B_r^{-1}\right\|}-\|f_s(k)\|\right) \\ &\quad +\left(\frac{u_0}{\left\|B_r^{-1}\right\|}-\|f_s(k)\|\right)^2 \end{aligned}$$

where $f_s(k)=-x_r(k)+x_d(k)+f_r(x(k))+d_r(k)-x_{r,d}(k+1)$, and if $\left\|B_r^{-1}\right\|\|f_s(k)\|\le u_0$ holds, then $\Delta V(k)\le 0$. ■

The third main result of this paper is the following.

Proposition 1. Given a desired output trajectory y_d, a dynamic system with output y, and a neural network with output $\widehat{y}$, the following inequality holds [5]:

$$\|y_d-y\|\le\|\widehat{y}-y\|+\|y_d-\widehat{y}\|$$

where y_d-y is the system output tracking error, and $\widehat{y}-y$ is the output estimation error and $y_d-\widehat{y}$ is the output tracking error of the nonlinear observer.

Based on this proposition, it is possible to divide the tracking objective into two parts [5]:
1. Minimization of $\widehat{y}-y$, which can be achieved by the proposed on-line nonlinear observer algorithm (11) as shown in Theorem 1.
2. Minimization of $y_d-\widehat{y}$. For this, a tracking algorithm is developed on the basis of the nonlinear observer (5). This minimization is obtained by designing the control law (40), as shown in Theorem 2.

VI. An Example

To illustrate the applicability of the proposed approach, we show the synchronization of two representative chaotic systems in this section.

Nonlinear systems have very rich dynamical behaviors including chaos, characterized by high sensitivity to parameter variation, external disturbance, and particularly tiny changes of initial conditions [2]. Chaotic attractors are globally bounded but locally unstable, which presents some important properties that have technological applications [2]. Two typical discrete chaotic systems are used for simulation here, on their synchronization under the control of the proposed scheme.

A. The Hénon System

The Hénon system is given by [1]

$$\begin{aligned} x_1(k+1) &= -ax_1^2(k)+x_2(k)+1 \\ x_2(k+1) &= bx_1(k) \\ y(k) &= x_1(k) \end{aligned} \tag{41}$$

where a and b are real parameters.

B. The Lozi System

The Lozi system is given by [3]

$$\begin{aligned} x_{1,d}(k+1) &= -p\left|x_{1,d}(k)\right|+x_{2,d}(k)+1 \\ x_{2,d}(k+1) &= qx_{1,d}(k) \end{aligned} \tag{42}$$

where p and q are real parameters.

C. Simulation Results

The Hénon chaotic attractor is generated by (41) with $a=1.4$, $b=0.3$, $\chi_1(0)=0.4$, $\chi_2(0)=0.4$. The goal is to force the chaotic Hénon attractor to synchronize to the Lozi attractor generated by (42) with $p=1.8$, $q=-1$, $\chi_1(0)=-5000$, $\chi_2(0)=5000$.

In order to estimate the state of (41), the following observer is used:

$$\begin{aligned} x_1(k+1) &= w_{11}(k)S(x_1(k))+w_{12}(k)x_2(k) \\ x_2(k+1) &= w_{21}(k)S^2(x_1(k)) \\ &\quad +w_{22}(k)S^2(x_2(k))+w_{23}(k)u(k) \\ \widehat{y}(k) &= x_1(k) \end{aligned}$$

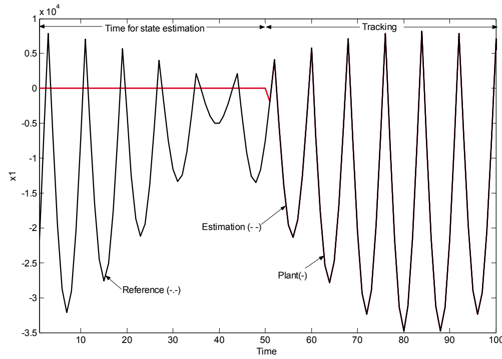

Fig. 1. Time evolution of the state $x_1(k)$ (solid line), its estimate $\widehat{x}_1(k)$ (dashed line) and its reference $x_{1,d}(k)$ (dash-dot line)

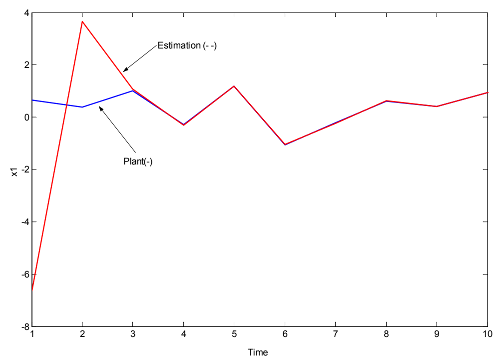

Fig. 2. Time evolution zoom for the state $x_1(k)$ (solid line), and its estimate $\widehat{x}_1(k)$ (dashed line)

which is in the block canonical form with the control law (40) and

$$
\begin{aligned}
u_{eq}(k) &= [w_{23}(k)]^{-1} \\
&\quad \times \left(-f_r(x(k)) + x_2^d(k+1)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
f_r(x(k)) &= w_{21}(k)\, S^2(x_1(k)) + w_{22}(k)\, S^2(x_2(k)) \\
x_2^d(k) &= [w_{12}(k)]^{-1}[-w_{11}(k)\, S(x_1(k)) + k_1 x_1(k)]
\end{aligned}
$$

The simulation results are shown in Figs. 1–7, where the state estimate evolution and the attractors are presented, which verify the theoretical analysis and design.

VII. Conclusions

In this paper, we have discussed a discrete-time output trajectory tracking by recurrent high-order neural network (RHONN) controller design and its application to chaos synchronization. First, a nonlinear observer is designed based on a RHONN trained with an EKF-based algorithm, where the training of the nonlinear observer is performed on-line in a parallel configuration.

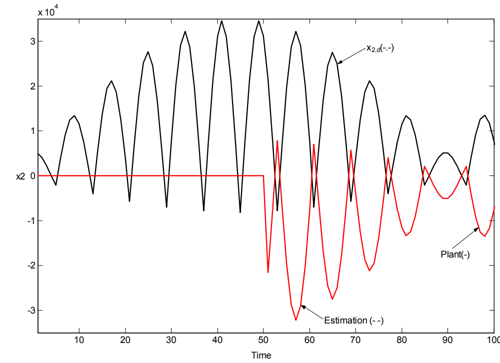

Fig. 3. Time evolution of the state $x_2(k)$ (solid line), its estimate $\widehat{x}_2(k)$ (dashed line) and $x_{2,d}(k)$ (dash-dot line)

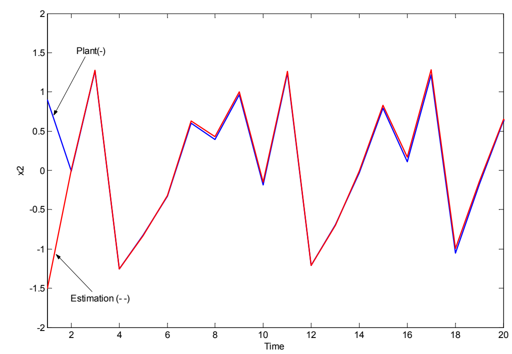

Fig. 4. Time evolution zoom for the state $x_2(k)$ (solid line), and its estimate $\widehat{x}_2(k)$ (dashed line)

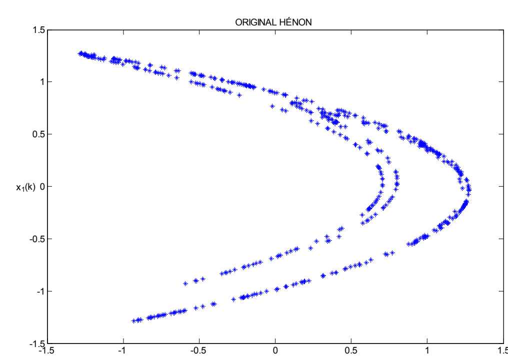

Fig. 5. Original Hénon attractor

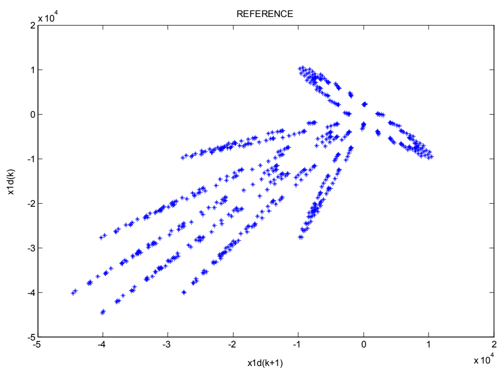

Fig. 6. Reference Lozi attractor

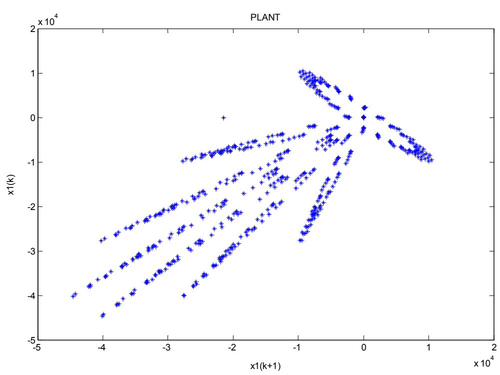

Fig. 7. Synchronized attractor

Then, using the designed control mechanism, synchronization of two chaotic systems is performed by means of the sliding-mode block control technique. Simulation results illustrate the applicability of the proposed control scheme.

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