

Sampling

- Most signals are either explicitly or implicitly sampled.
- Sampling has both similarities and differences in the 1-D and 2-D cases.



1-D Sampling Theorem (Review)

- A bandlimited signal can be periodically sampled with no loss of information provided it is sampled often enough.
- The minimum sampling rate (Nyquist) rate is proportional to the signal's bandwidth.
- The DTFT of the samples can be related to the Fourier transform of the continuous signal that is sampled.

Sampling

- Sampling in one domain \leftrightarrow Periodicity in other
 - \Rightarrow Replicas may overlap and we lose signal (Aliasing)
 - \Rightarrow If no overlap, \Rightarrow we can recover signal by keeping one replica

Sampling 2D signals

- There are many ways to sample 2D signals



Analog Image



Sampling example



Sampling example

Sampling 2D signals

- The simplest and most common of this is rectangular sampling

Sample t_1 and t_2

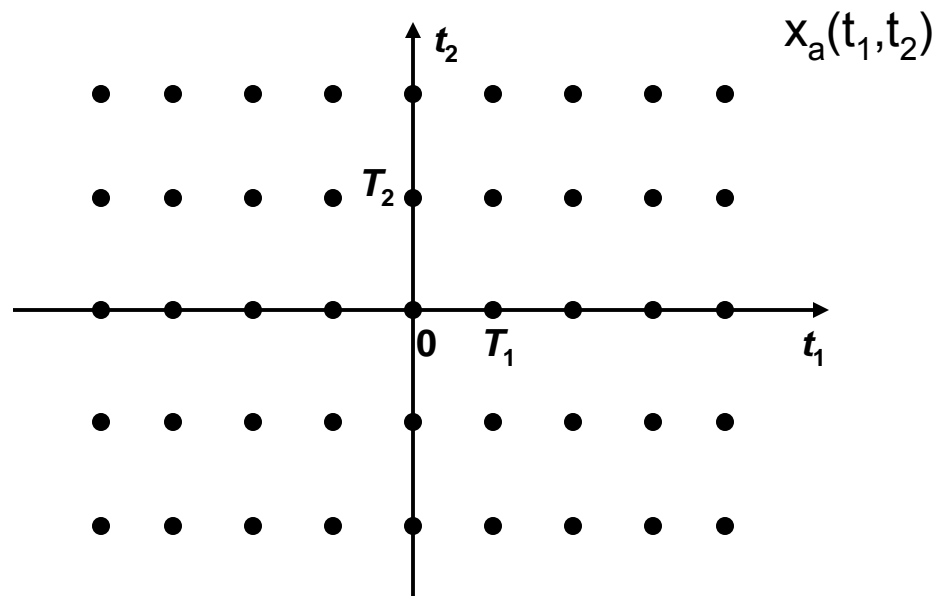
$$\mathbf{x}_a(\mathbf{t}_1, \mathbf{t}_2) \longleftrightarrow \mathbf{x}_a(n_1 T_1, n_2 T_2) = \mathbf{x}(n_1, n_2)$$

Continuous-domain 2D signal discrete time at regular intervals indices of samples

- T_1 : Horizontal sampling interval \Rightarrow F.T. replicated with period $2\pi/T_1$ in horizontal direction
- T_2 : Vertical sampling interval \Rightarrow F.T. replicated with period $2\pi/T_2$ in vertical direction

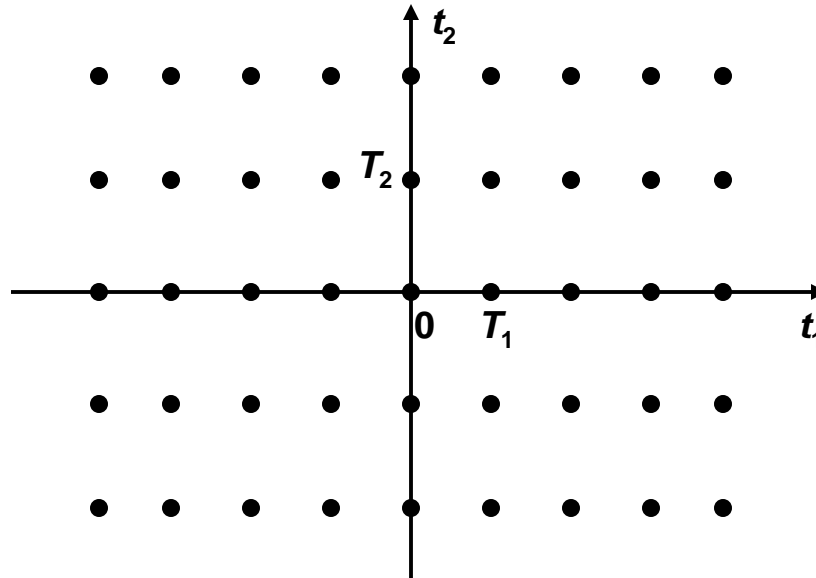
Sampling 2D signals

- The simplest and most common of this is rectangular sampling



Derivation (Sampling Theorem)

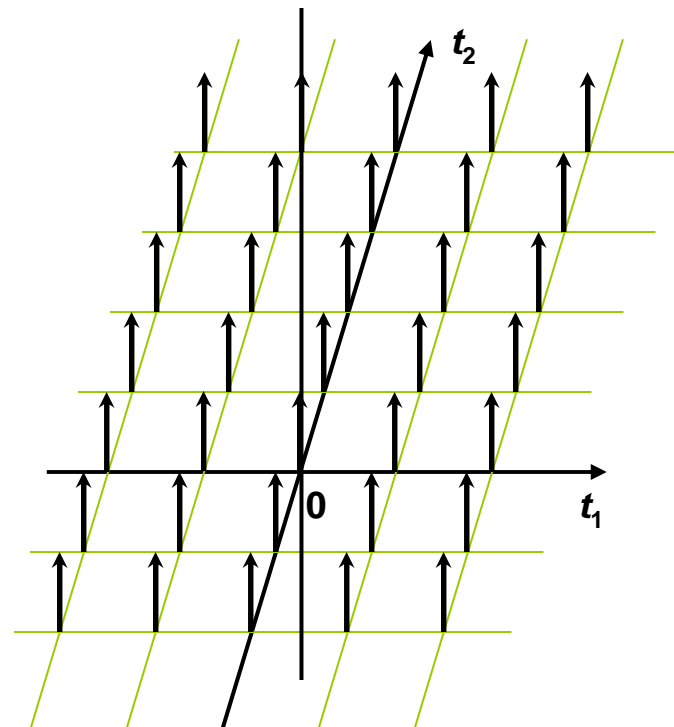
- Define $p_a(t_1, t_2)$ to be a field of impulses- one at each sampling location (bed of nails)



Sampling 2D signals

- The Bed of Nails

$$p_a(t_1, t_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(t_1 - n_1 T_1, t_2 - n_2 T_2)$$



Sampling 2-D signals

- **2-D continuous Fourier transform:**

$$X_a(\Omega_1, \Omega_2) \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_a(t_1, t_2) \exp[-j\Omega_1 t_1 - j\Omega_2 t_2] dt_1 dt_2$$

$$x_a(t_1, t_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_a(\Omega_1, \Omega_2) \exp[j\Omega_1 t_1 + j\Omega_2 t_2] d\Omega_1 d\Omega_2$$

- **Relevant properties:**

$$x_a(t_1, t_2) ** h_a(t_1, t_2) \leftrightarrow X_a(\Omega_1, \Omega_2) \cdot H(\Omega_1, \Omega_2)$$

$$x_a(t_1, t_2) \cdot p_a(t_1, t_2) \leftrightarrow \frac{1}{4\pi^2} X_a(\Omega_1, \Omega_2) ** P_a(\Omega_1, \Omega_2)$$

Sampling 2-D signals

- The bed-of-nails (field of impulses):

$$p_a(t_1, t_2) \triangleq \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(t_1 - n_1 T_1, t_2 - n_2 T_2)$$

- The Fourier transform of a bed-of-nails is another bed-of-nails with reciprocal spacing

$$p_a(\Omega_1, \Omega_2) = \frac{(2\pi)^2}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \delta\left(\Omega_1 - \frac{2\pi k_1}{T_1}, \Omega_2 - \frac{2\pi k_2}{T_2}\right)$$

Sampling 2D signals

- $x_a(t_1, t_2)$ can be *rectangularly* sampled as follows:

$$x_s(t_1, t_2) = x_a(t_1, t_2) \cdot p_a(t_1, t_2)$$

$$x_s(t_1, t_2) = x_a(t_1, t_2) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(t_1 - n_1 T_1, t_2 - n_2 T_2) \quad (\star)$$

Sampled
but still
continuous

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_a(t_1, t_2) \delta(t_1 - n_1 T_1, t_2 - n_2 T_2)$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \underbrace{x_a(n_1 T_1, n_2 T_2)}_{x(n_1, n_2)} \delta(t_1 - n_1 T_1, t_2 - n_2 T_2) \quad (\star\star)$$

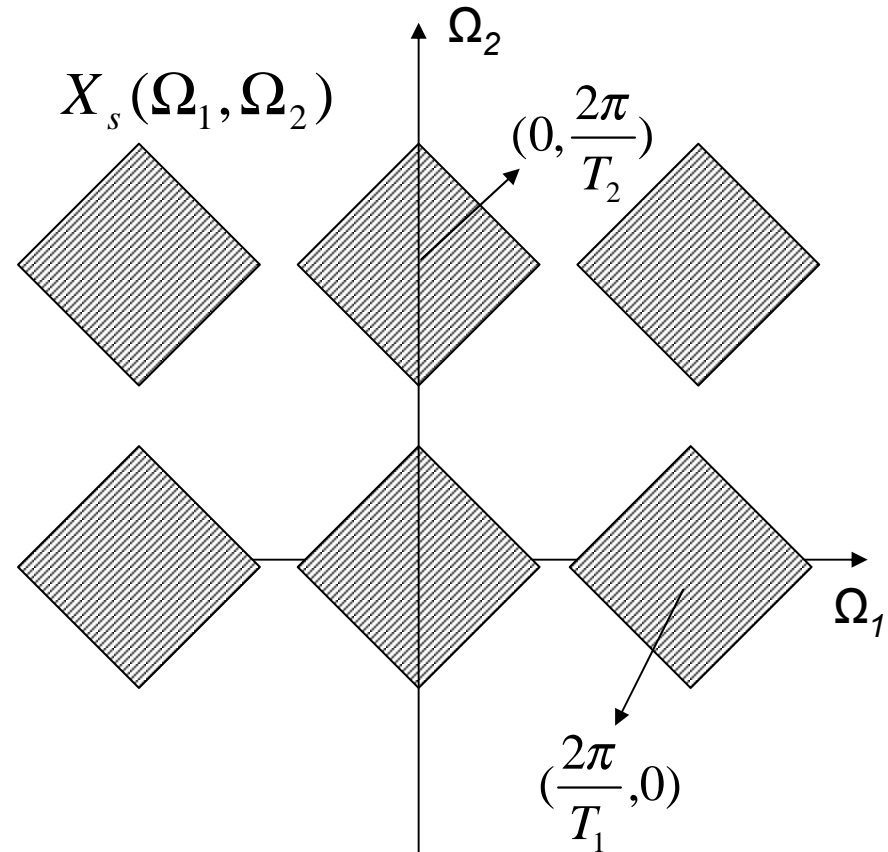
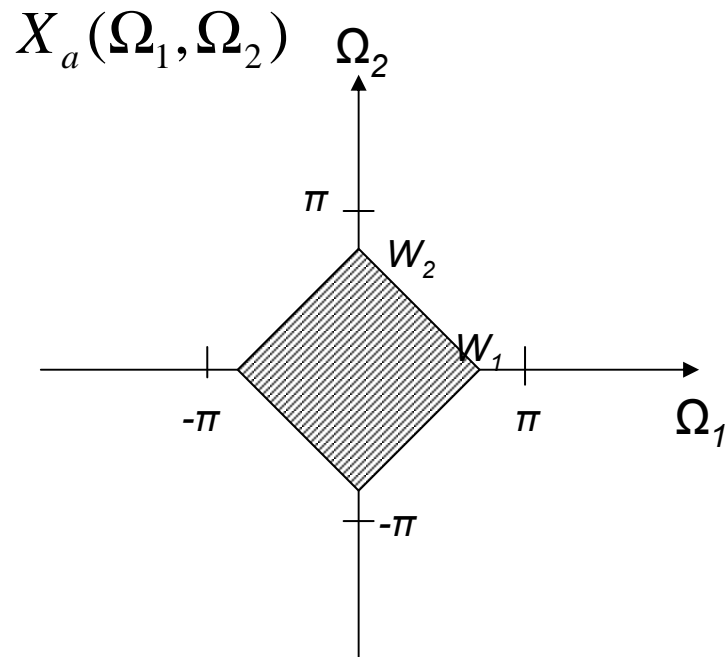
Sampling 2D signals

- The Fourier transform is

$$\begin{aligned} X_s(\Omega_1, \Omega_2) &= \frac{1}{4\pi^2} X_a(\Omega_1, \Omega_2) ** \frac{4\pi^2}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \delta\left(\Omega_1 - \frac{2\pi k_1}{T_1}, \Omega_2 - \frac{2\pi k_2}{T_2}\right) \\ &= \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a(\Omega_1, \Omega_2) ** \delta\left(\Omega_1 - \frac{2\pi k_1}{T_1}, \Omega_2 - \frac{2\pi k_2}{T_2}\right) \\ &= \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a\left(\Omega_1 - \frac{2\pi k_1}{T_1}, \Omega_2 - \frac{2\pi k_2}{T_2}\right) \end{aligned}$$

Sampling 2D signals

- In the frequency domain:



$\frac{2\pi}{T_1}$: Horizontal Period

$\frac{2\pi}{T_2}$: Vertical Period

Sampling 2D signals

- Let W_1 and W_2 be the horizontal and vertical bandwidth of $x_a(t_1, t_2)$
- Then exact recovery requires

$$\left. \begin{array}{l} T_1 < \frac{\pi}{W_1} \\ T_2 < \frac{\pi}{W_2} \end{array} \right\} \Rightarrow \text{no aliasing condition}$$

or, equivalently,

$$\left. \begin{array}{l} \Omega_{s1} = \frac{2\pi}{T_1} > 2W_1 \\ \Omega_{s2} = \frac{2\pi}{T_2} > 2W_2 \end{array} \right\} \Rightarrow \text{no aliasing condition}$$

or, equivalently,

$$\left. \begin{array}{l} f_{s1} = \frac{1}{T_1} > \frac{W_1}{\pi} \\ f_{s2} = \frac{1}{T_2} > \frac{W_2}{\pi} \end{array} \right\} \Rightarrow \text{no aliasing condition}$$

Relating the Discrete and Continuous Frequency Domains

- Recall:

$$x_s(t_1, t_2) = x_a(t_1, t_2) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(t_1 - n_1 T_1, t_2 - n_2 T_2) \quad (\star)$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \underbrace{x_a(n_1 T_1, n_2 T_2)}_{x(n_1, n_2)} \delta(t_1 - n_1 T_1, t_2 - n_2 T_2) \quad (\star\star)$$

- Taking the F.T. of (\star) :

$$X_s(\Omega_1, \Omega_2) = \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a\left(\Omega_1 - \frac{2\pi k_1}{T_1}, \Omega_2 - \frac{2\pi k_2}{T_2}\right)$$

- Taking the F.T. of $(\star\star)$:

$$\begin{aligned} X_s(\Omega_1, \Omega_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n_1} \sum_{n_2} x(n_1, n_2) \delta(t_1 - n_1 T_1, t_2 - n_2 T_2) e^{-jt_1 \Omega_1} e^{-jt_2 \Omega_2} dt_1 dt_2 \\ &= \sum_{n_1} \sum_{n_2} x(n_1, n_2) e^{-jn_1 T_1 \Omega_1} e^{-jn_2 T_2 \Omega_2} = X(\underbrace{\Omega_1 T_1}_{\omega_1}, \underbrace{\Omega_2 T_2}_{\omega_2}) \end{aligned}$$

where

$$X(\omega_1, \omega_2) = \text{DTFT}\{x(n_1, n_2)\} = \sum_{n_1} \sum_{n_2} x(n_1, n_2) e^{-jn_1 \omega_1} e^{-jn_2 \omega_2}$$

Relating the Discrete and Continuous Frequency Domains

- Comparing, one can easily see that:

$$X(\omega_1, \omega_2) = X_s\left(\frac{\omega_1}{T_1}, \frac{\omega_2}{T_2}\right) = \frac{1}{T_1 T_2} \sum_{k_1} \sum_{k_2} X_a\left(\frac{\omega_1}{T_1} - \frac{2\pi k_1}{T_1}, \frac{\omega_2}{T_2} - \frac{2\pi k_2}{T_2}\right)$$

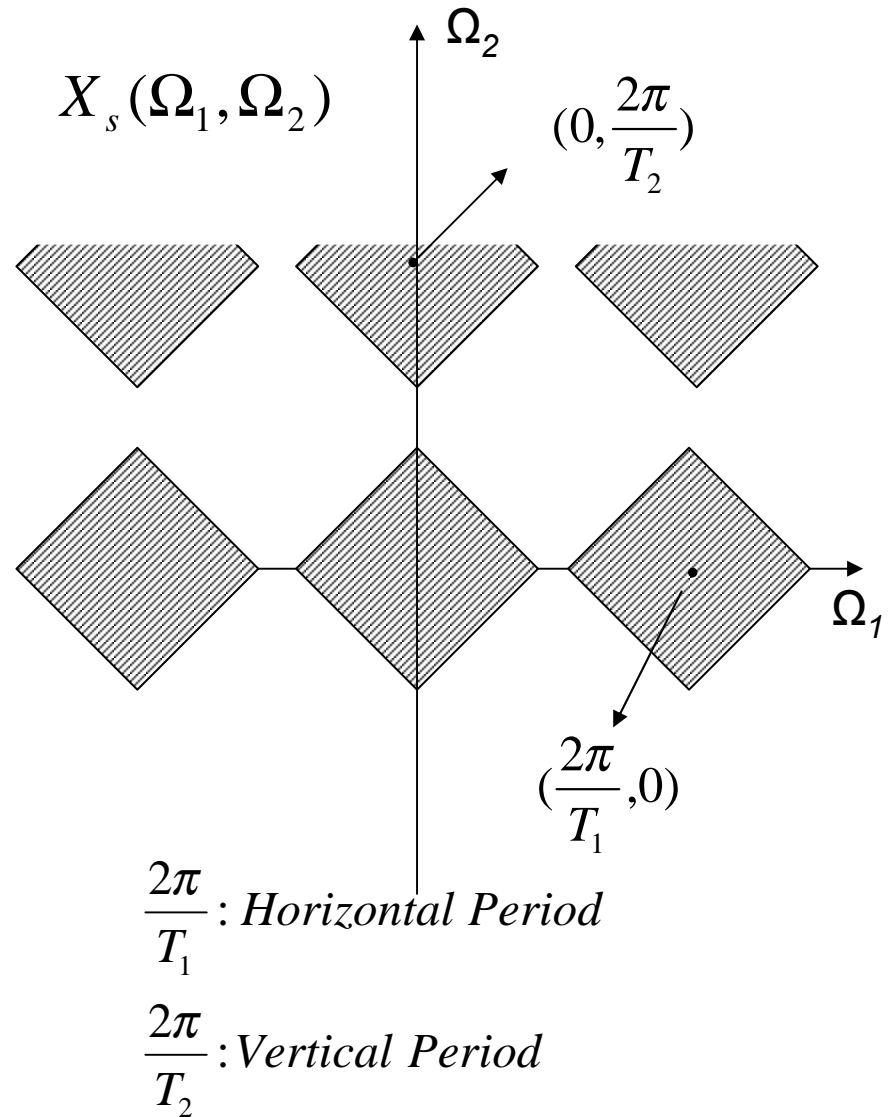
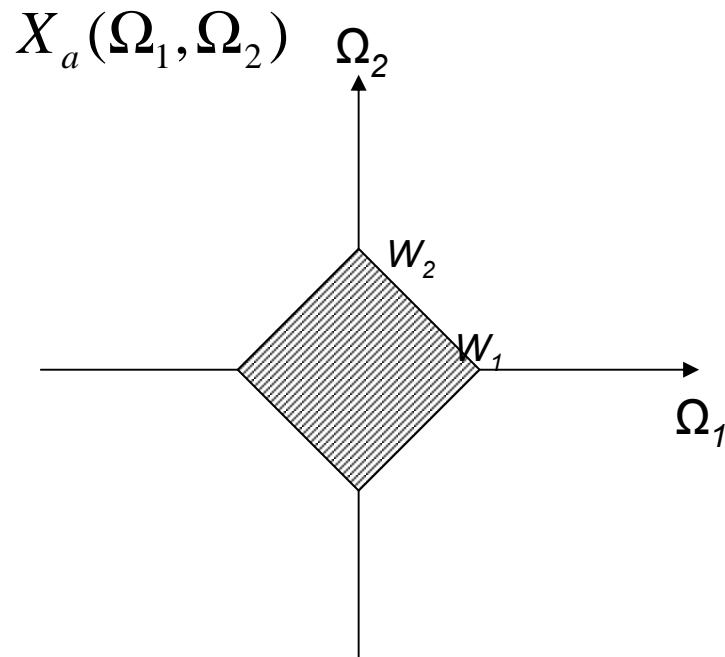
- As in the 1-D case, we have a normalization relation between $(\omega_1$ and $\Omega_1)$ and $(\omega_2$ and $\Omega_2)$ given by:

$$\begin{array}{ccc} \text{radians} & \begin{array}{l} \nearrow \omega_1 = \Omega_1 T_1 \\ \searrow \omega_2 = \Omega_2 T_2 \end{array} & \text{or} \\ & & \begin{array}{l} \nearrow \Omega_1 = \omega_1 / T_1 \\ \searrow \Omega_2 = \omega_2 / T_2 \end{array} \\ & & \text{radians/sec} \end{array}$$

- If $x(n_1 n_2) = x_a(n_1 T_1, n_2 T_2)$, then $X(\omega_1, \omega_2) = \text{DTFT}\{x(n_1 n_2)\}$ is related to $X_a(\Omega_1, \Omega_2) = \text{FT}\{x_a(t_1, t_2)\}$ by:
 - Periodicity (aliasing, repetition)
 - Amplitude scaling
 - Frequency normalization

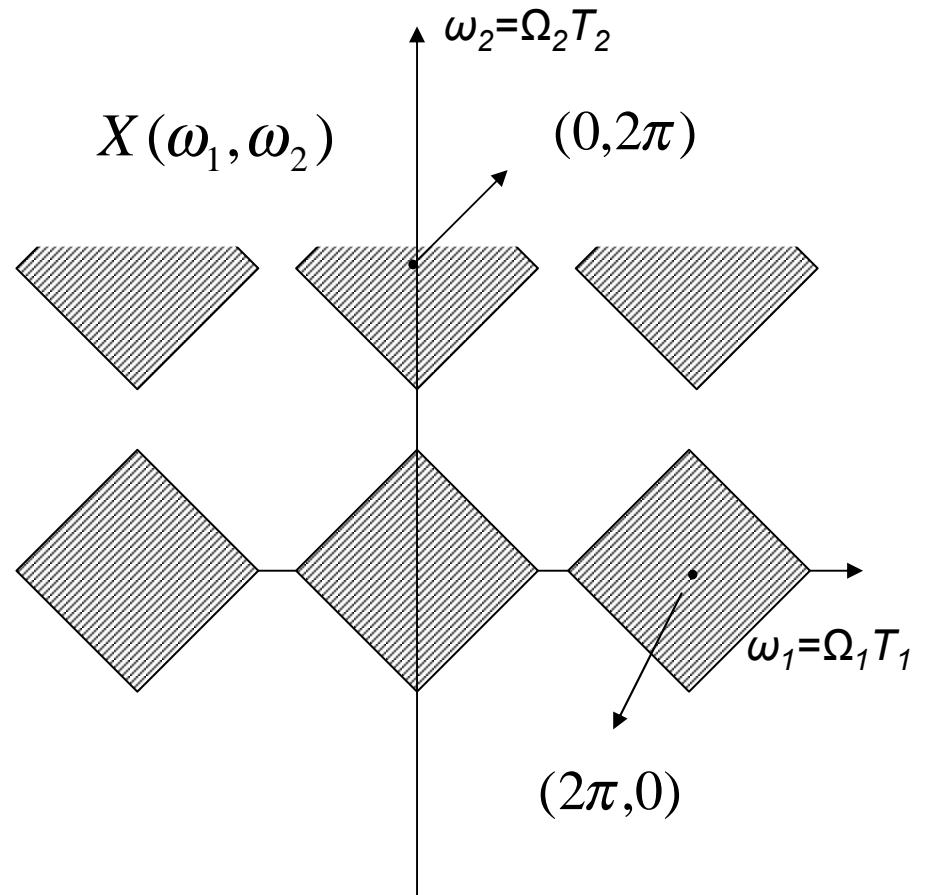
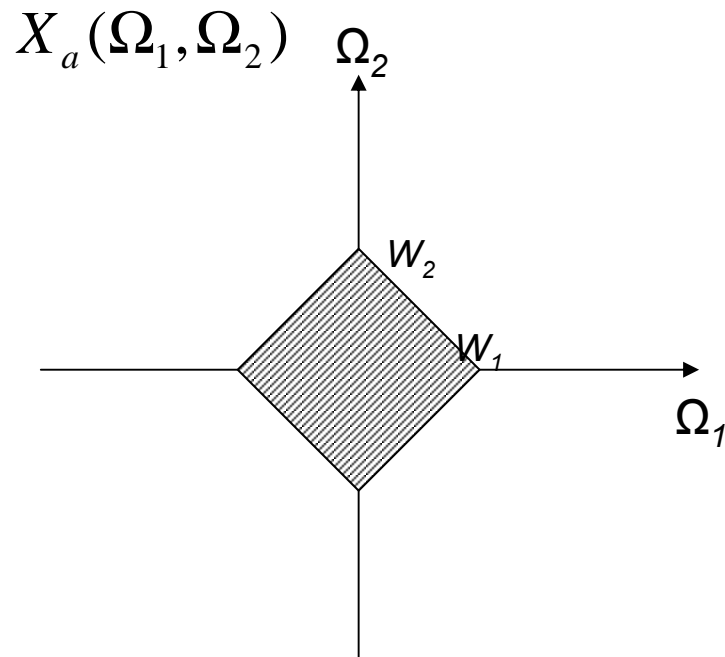
Sampling 2D signals

- In the frequency domain:



Sampling 2D signals

- In the frequency domain:



2π : Horizontal Period

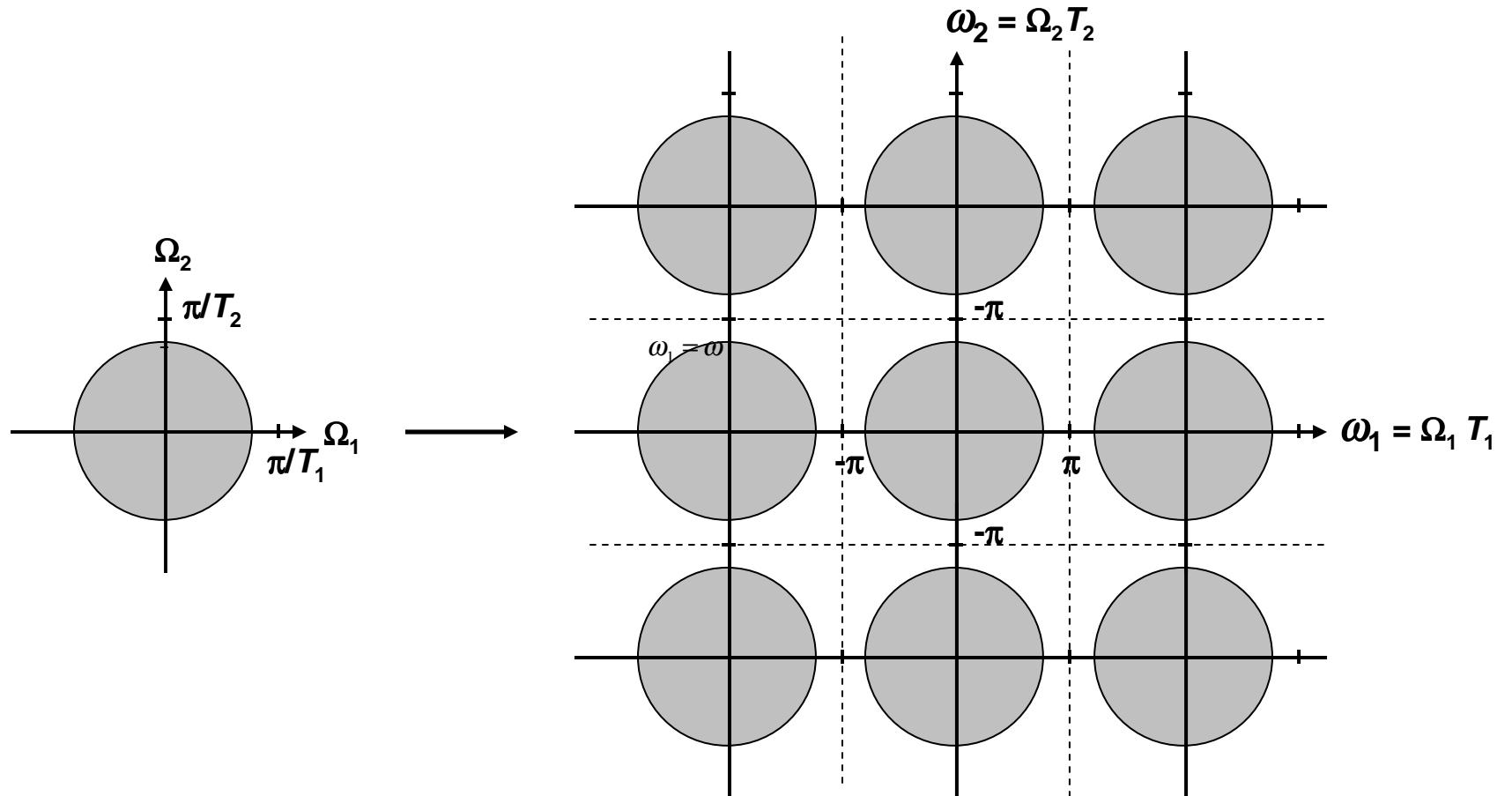
2π : Vertical Period

Sampling 2D signals

- Aliasing occurs if we sample below the Nyquist rate
 - critical sampling \Rightarrow rate = Nyquist rate \Rightarrow no aliasing
 - oversampling \Rightarrow rate $>$ Nyquist rate \Rightarrow no aliasing
 - ✓ usually oversample since nonideal antialiasing filter (does not have to have sharp cutoff) \Rightarrow replicas more spaced out
 - undersampling \Rightarrow rate $<$ Nyquist rate \Rightarrow Aliasing \Rightarrow cannot recover signal

Sampling 2D signals

- Oversampling Example: no aliasing



Sampling 2D signals

- Undersampling Example: undersampling gives aliasing:

